Geometry of VAS Reachability Sets

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Technical University of Munich

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Introduction to Vector Addition Systems

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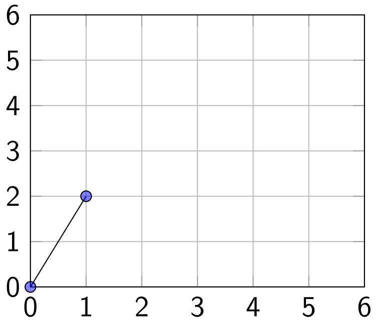

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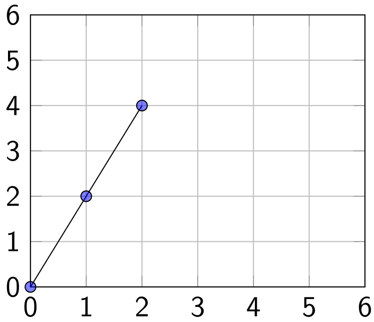

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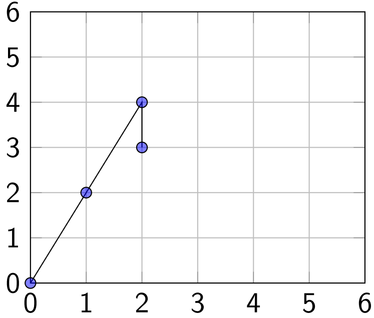

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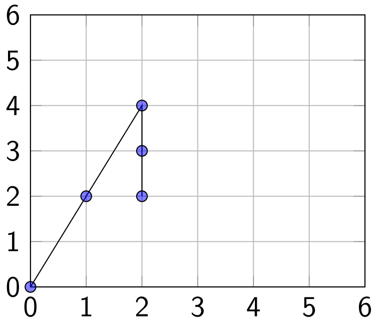

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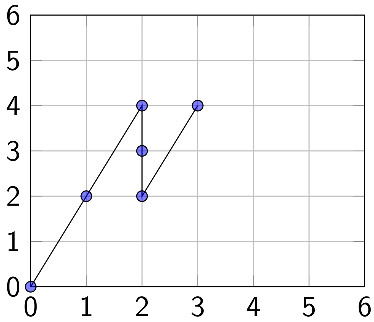

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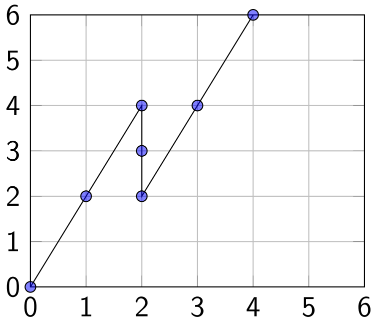

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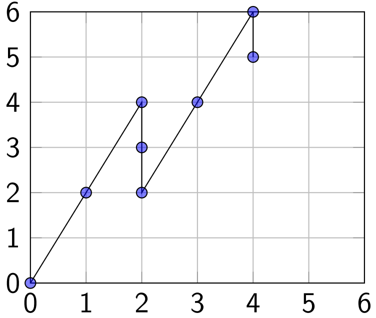

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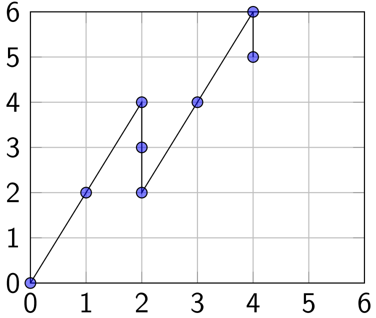

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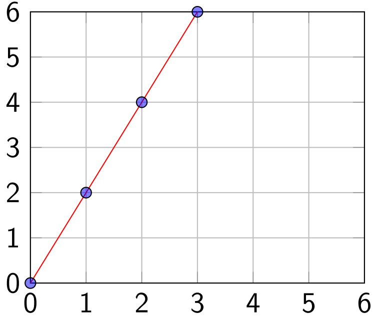

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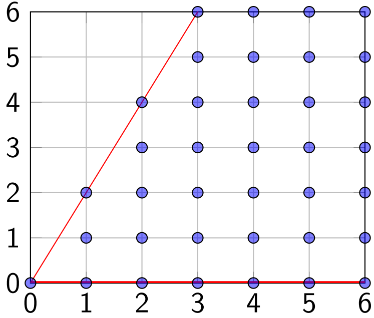

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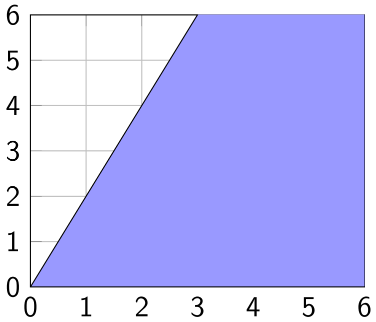
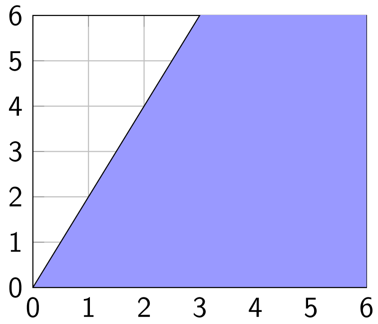


Semilinear Sets

Semilinear Sets

Monoid: Cone-like object



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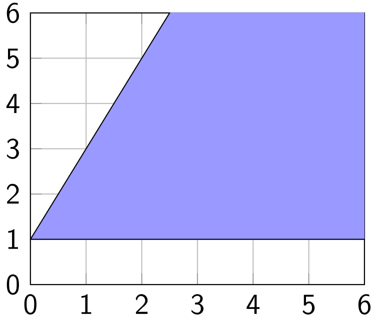
Linear set = shifted monoid



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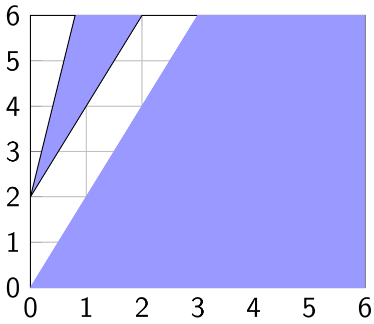
Semilinear set: Finite union of linear sets



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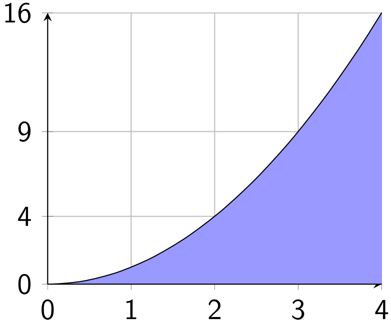

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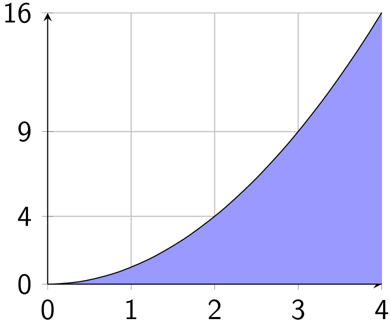

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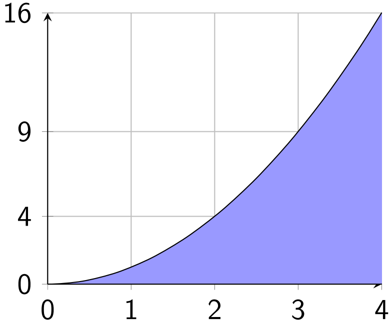

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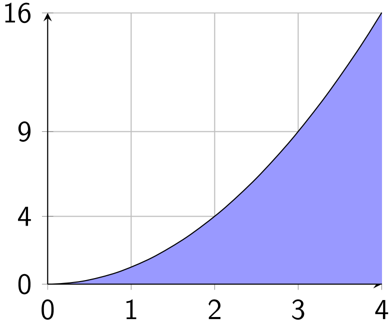

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Goal: Is there some class of sets which

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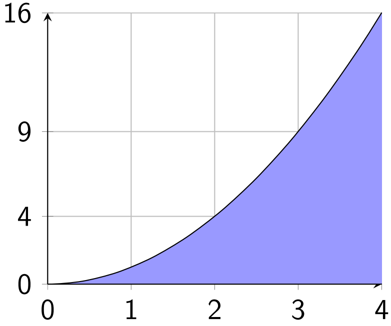
- is easy to deal with?



Prior Work, Pseudo-Linear

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The General Vector Addition System Reachability Problem by Presburger Inductive Invariants

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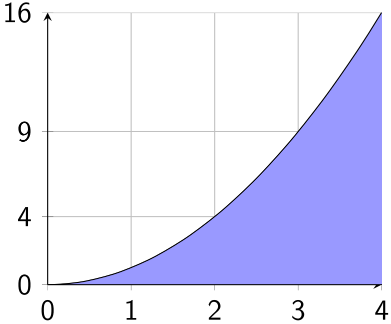
By Jerome Leroux, LICS, 2009

Prior Work, Pseudo-Linear

Set R,



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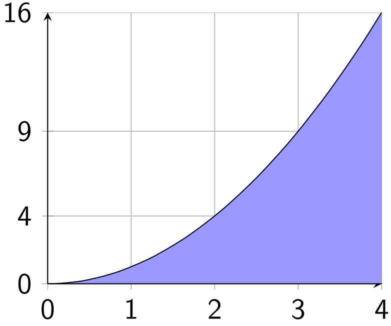

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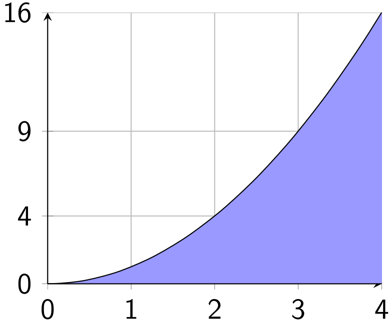

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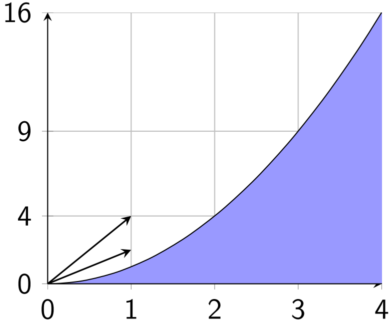

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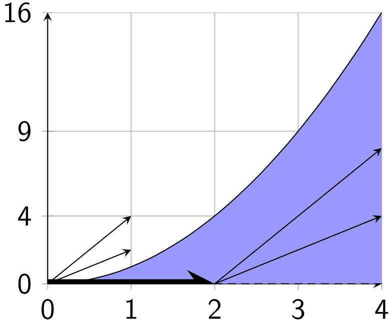

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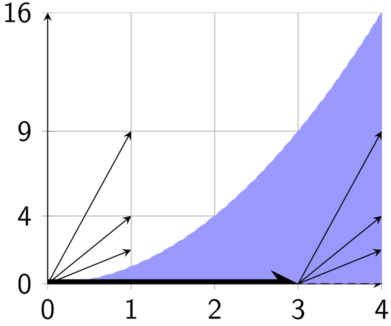

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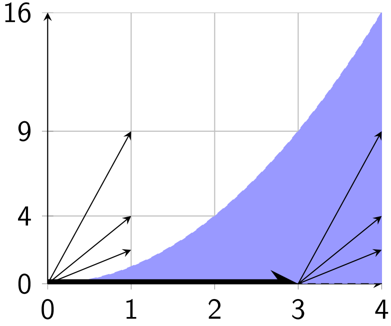

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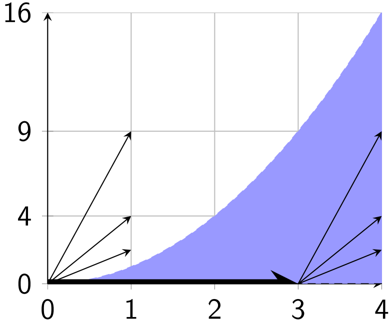

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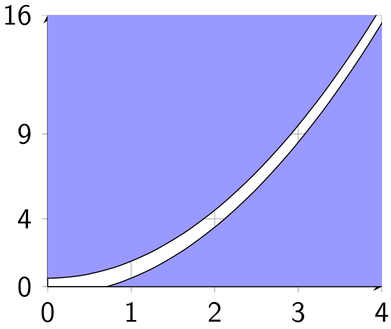

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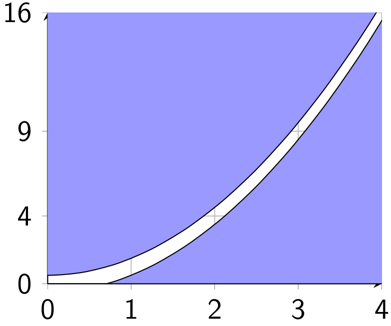

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Reachability sets are semi-pseudo-linear

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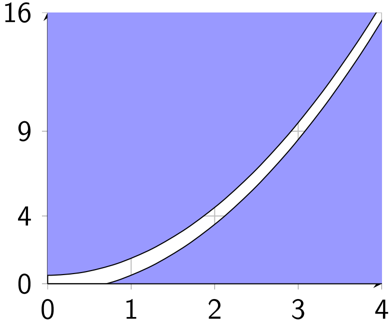

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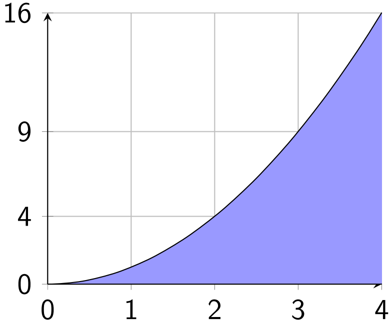

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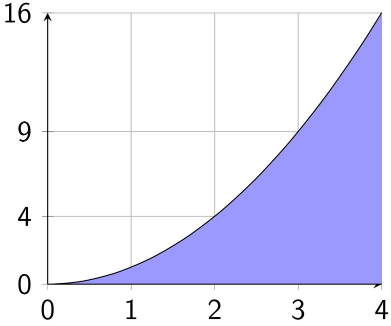

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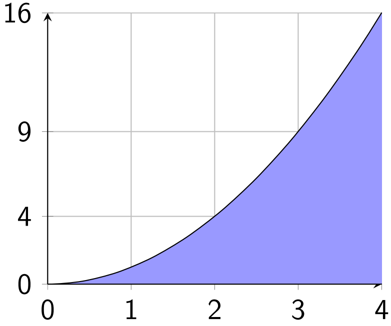

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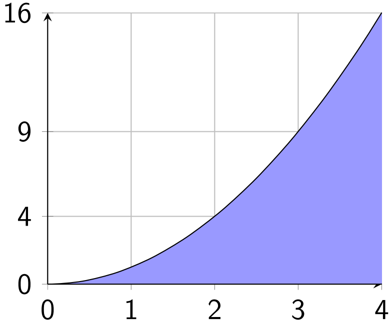

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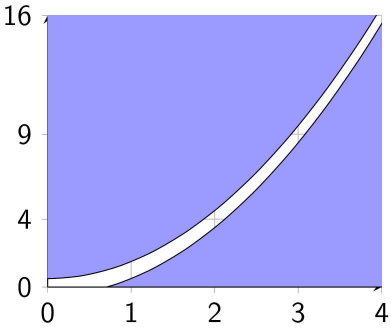

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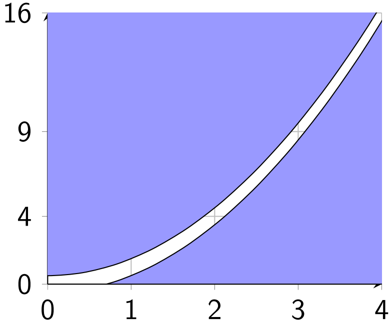
Semi-pseudo-linear as a class



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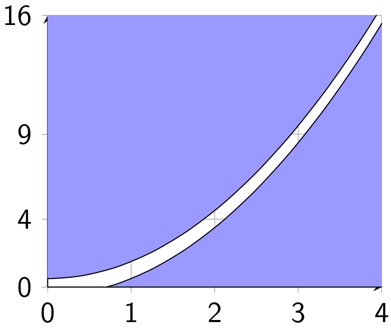

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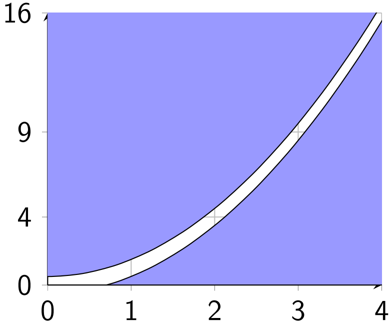

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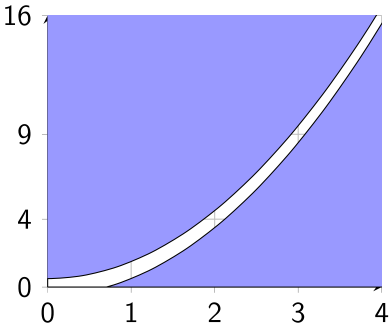
Find class between reachability sets and
semi-pseudo-linear.



New-Pseudo-Linear

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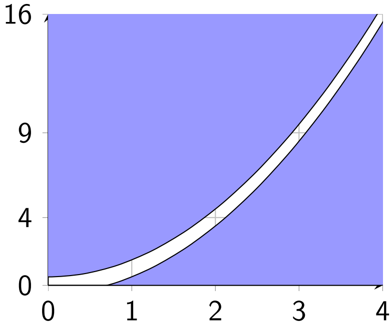

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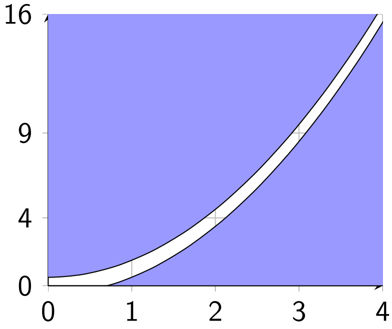

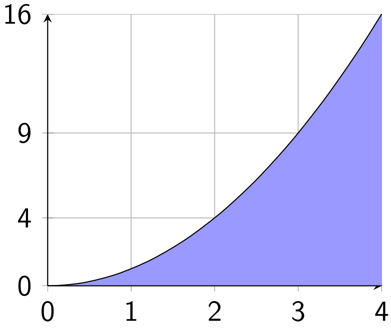

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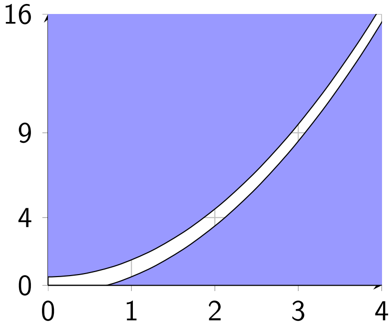

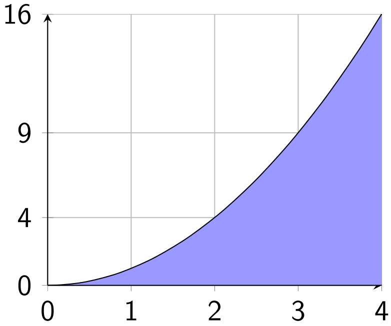

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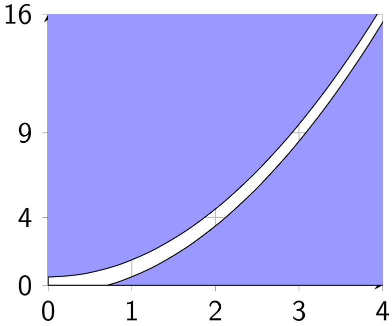

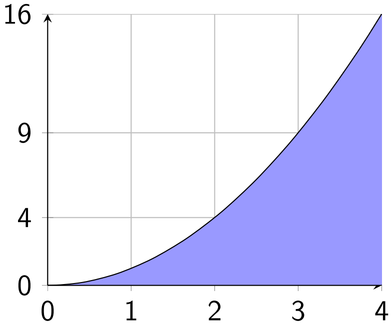

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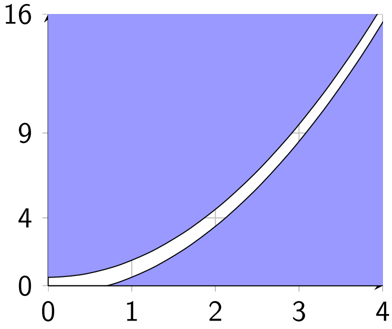

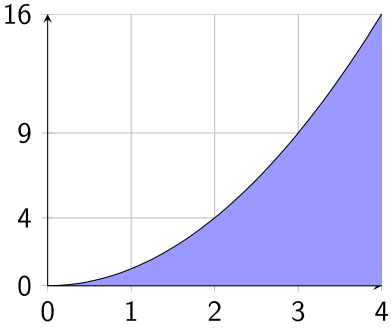

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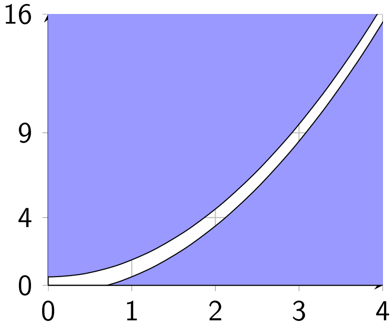
for every finite set F of directions interior to L

exists v s.t. $v + Monoid(F) \subseteq R$



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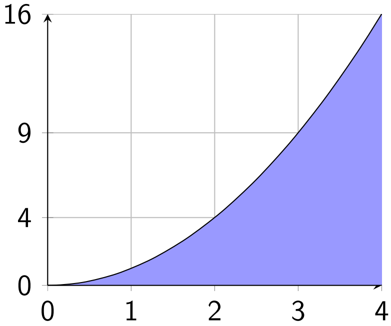

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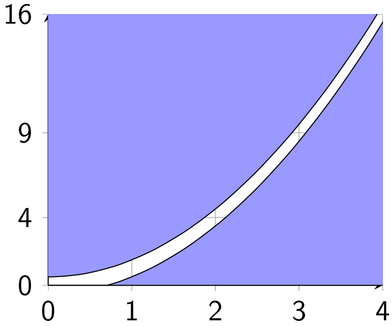

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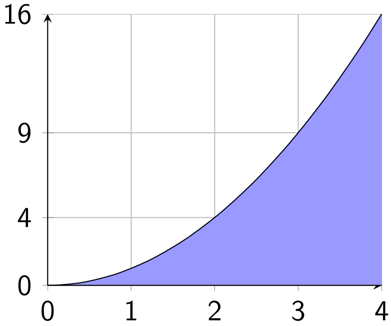

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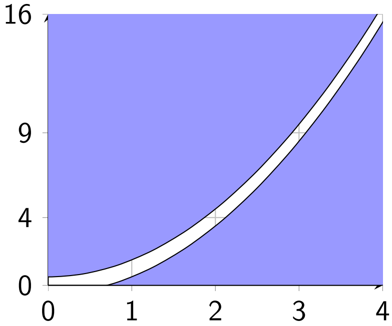

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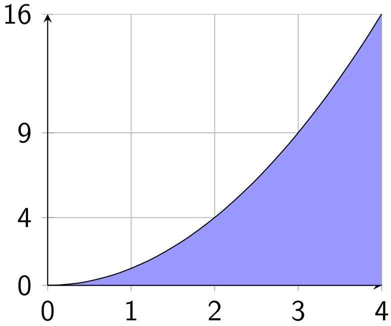

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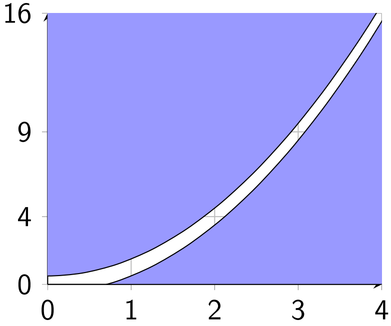

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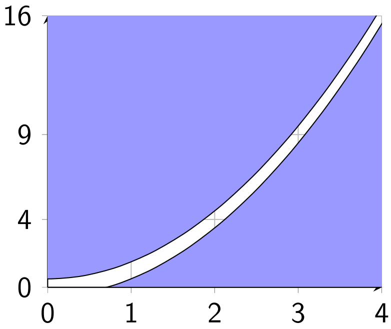

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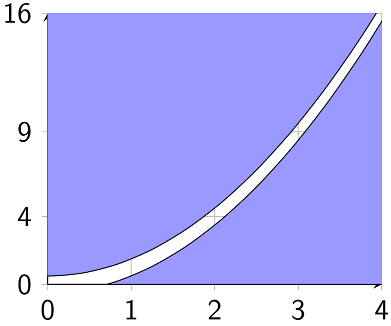

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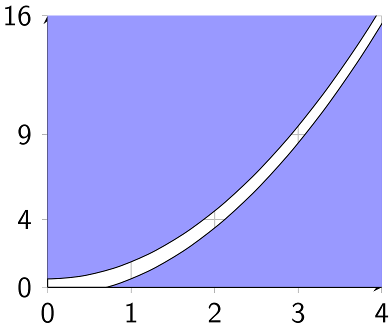

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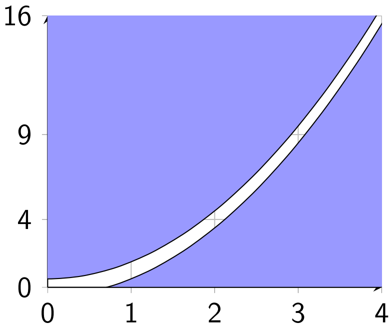

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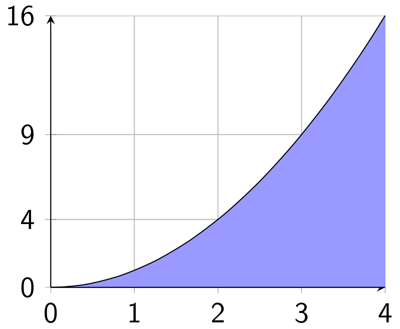

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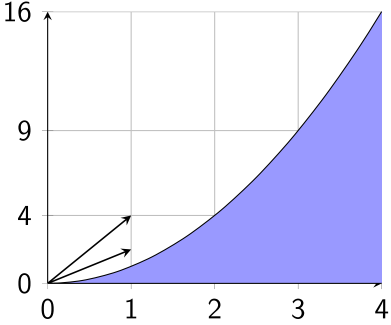

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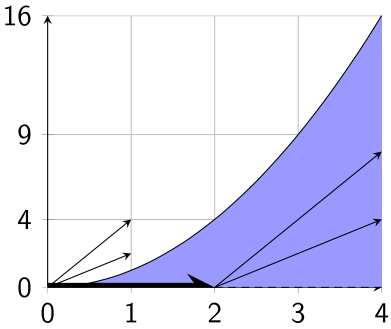

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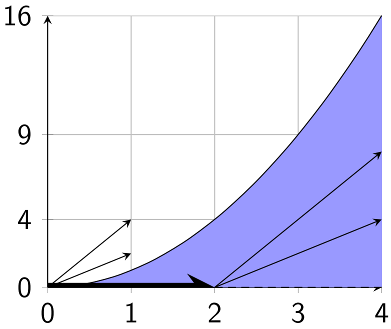

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New-semi-pseudo-linear: Finite unions.

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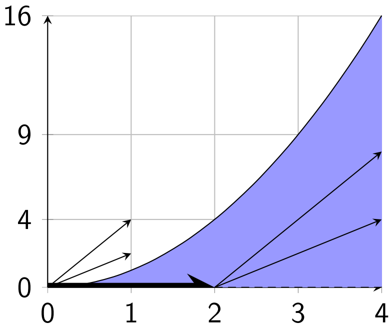

Our Results

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Theorem (Utilizing KLMST Decomposition)

Let R reachability set, L linear set. Then $R \cap L$ is new-semi-pseudo-linear.

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Theorem (Used to be Conjecture)

Let R be a reachability set. Then for every linear L,
if $|L \setminus R| = \infty$, then there exists an infinite linear set $\subseteq (L \setminus R)$.

Thank you for your attention!

Are there questions?